Conference on Bluff Body Wakes and Vortex-Induced Vibrations (BBVIV3)
Port Douglas, Australia
17-20 December, 2002

A Re-examination of the Stuart-Landau Model Applied to Cylinder Wake Transition

Mark C Thompson[a] and Patrice Le Gal[b]

[a] *Fluids Laboratory for Aeronautical and Industrial Research (FLAIR)*
Department of Mechanical Engineering
Monash University, Melbourne, 3800, Australia

[b] *Institut de Recherche sur les Phénomènes Hors Equilibre*
9 Rue F. Joliot-Curie, BP. 146, 13384, Marseille, Cedex 13, France

Following previous experimental and computational studies, we further investigate the applicability of the Stuart-Landau equation to describe the Hopf bifurcation occurring for flow past a circular cylinder. In contrast to previous findings, it is shown that when the amplitude variable is taken as the transverse velocity component at a point in the wake, the so-called Landau constant varies considerably with position and importantly is generally far from constant during the saturation phase of wake development. However, it is found that the Landau constant at saturation is indeed a position-independent constant and this value is close to that generally measured previously both experimentally and numerically.

1. INTRODUCTION

The complex Stuart-Landau equation has been widely used to model supercritical bifurcations occurring in flow systems when a control parameter exceeds a critical value. Typical examples include: the transition from steady flow to vortex shedding (*i.e.*, the Hopf bifurcation) in the wake of a circular cylinder (Dusek *et al.* 1994, Sreenivasan *et al.* 1986, Provansal *et al.* 1987, Schumm *et al.* 1994, Albarède & Provansal 1995, Zielinska & Wesfried 1995); the regular bifurcation (*i.e.*, steady to steady) of a sphere wake leading to the beautiful two-tailed structure as shown by Margarvey & Bishop (1961) (*e.g.*, Thompson *et al.* 2000, Ormières & Provansal 1999); the Hopf bifurcation of the sphere wake (Thompson *et al.* 2000, Provansal & Ormières 1998, Ghidersa & Dusek 2001); and the transition to three-dimensional *mode B* shedding of a two-dimensional circular cylinder wake (Henderson 1997). However, not all flow transitions are governed by the cubic form of the Stuart-Landau model, for instance, the initial three-dimensional transition of a circular cylinder wake from the two-dimensional Bénard-von Karman wake is *subcritical* and hence requires the retension of at least quintic terms to model the approach to the saturated state (Henderson 1997). These successes have been achieved despite an incomplete mathematical foundation for the Stuart-Landau model.

A key reason for studying the (Stuart-) Landau model is that, because it is amenable to straight-forward mathematical analysis, it can predict the behaviour of, and provide important insight into, complex flow systems. For instance, Landau models can be coupled together to describe the wake dynamics of interacting cylinder wakes (Peschard, 1995). The model also provides a starting point for the Ginzburg-Landau model describing aspects of the two-dimensional shedding from a circular cylinder, such as phase transitions, oblique shedding and chevron patterns (Albaredé & Provansal 1995, Williamson 1988). The model can be extended to describe interacting wake modes such as the two initial three-dimensional circular cylinder wake modes (Barkley *et al.* 1999). In theory, it can also be extended to wake flows from forced or freely-oscillating bodies. For example, it was applied to predict the asymptotic wake states of a circular cylinder wake under transverse forcing by Le Gal *et al.* (2000). For forced flows a complete mathematical analysis of the different possibilities has been provided by Gambaudo (1985).

The aim of the present paper is to re-examine the application of the Stuart-Landau model to the Hopf bifurcation of the circular cylinder wake. The work of Dusek *et al.* (1994) showed that the model accurately describes the observed response when the Reynolds number is restricted to be within 10% of the critical value. These authors used the transverse velocity on the wake centreline as the Landau model variable. For points on the wake centreline, the transverse velocity is zero prior to transition, and hence this provides a direct measure of the growth of the instability. Importantly they found that the *Landau constant* indeed appeared to be a constant at all sampled points in the wake.

2. THEORY

2.1. THE STUART-LANDAU EQUATION

The complex Stuart-Landau equation is given by

$$\frac{dA}{dt} = (\gamma + i\omega)A - (c_R + ic_I)|A|^2A + \ldots, \tag{1}$$

in which A is a complex-valued function of time t and the parameters γ, ω, c_R and c_I are all real. The Landau constant, usually denoted by c, is given by $c = c_I/c_R$ in this formulation. The equation is generally truncated after the cubic term as is the usual case for supercritical transitions since the cubic term is nominally sufficient for limiting the initial exponential growth and causing saturation. This is the case for the Hopf bifurcation, *i.e.*, the transition to periodic shedding in the circular cylinder wake. Importantly, the real part of the cubic coefficient is positive so that this term is responsible for saturation. Also note that only odd terms in the (complex) amplitude can appear on the righthand side.

Equation (1) represents the normal form of the Hopf bifurcation which occurs at the critical value of the parameter $\gamma = 0$. For $\gamma < 0$, the null solution $A = 0$ is a stable solution. For a circular cylinder, the flow corresponds to steady flow with attached eddies at the rear of the cylinder. For $\gamma > 0$, this base state loses its stability and the solution settles down to a time-periodic state (corresponding to Bérnard–von Karman vortex shedding). If only the linear and cubic terms are considered, the saturation amplitude is given by $|A| = (\gamma/c_R)^{1/2}$, and the angular frequency at saturation is given by $\omega - \gamma c$. The time-scale for the transient approach to this final periodic state is given by γ^{-1} (*e.g.*, Dusek *et al.* 1994).

In general, the parameters in this equation may be a function of Reynolds number although it is hoped that the dependence is weak, except for γ which changes from negative to positive at transition. Also, if A is taken as a local quantity, such as the transverse velocity component, the coefficients may be function of position. It is found that γ and ω are independent of position close to transition, this can be justified theoretically and from measurements. For the analysis presented here we take the coefficients c_R and c_I to depend on $|A|^2$. This is equivalent to including higher-order terms in the equation but allows us instead to focus on the constancy of these parameters during the exponential growth and saturation phases of the transition. To reiterate, for the analysis presented in this article we explicitly truncate equation (1) to include only linear and cubic terms. The effect of higher-order terms is then implicitly included by allowing the real and imaginary cubic coefficients to depend on $|A|^2$. Note that in terms of this truncated model, the real and imaginary cubic coefficients at zero amplitude correspond to the cubic coefficients of the original (untruncated) expansion.

To proceed with the analysis of the Landau model, it helps to write $A(t)$ in the form

$$A(t) = \rho(t)e^{i\phi(t)}, \tag{2}$$

where $\rho(t) = |A(t)|$ is the real and non-negative amplitude of the complex function A, and $\phi(t)$ is its phase (also real). Substitution into equation (1) results in the pair of equations

$$\frac{d\log\rho}{dt} = \gamma - c_R(\rho^2, \zeta)\rho^2 \tag{3}$$

and

$$\dot{\phi} = \omega - c_I(\rho^2, \zeta)\rho^2. \tag{4}$$

Again note that the cubic coefficients are now written as explicit functions of the time-dependent amplitude ($\rho = |A|$). They are also a function of position ($\zeta = x/R$) if we use the transverse velocity component as the Landau complex amplitude variable.

3. RESULTS

3.1. NUMERICAL METHODOLOGY

We describe a series of simulations which extend the work of Dusek *et al.* (1994). The spectral-element method is used to simulate the flow at post-critical Reynolds numbers. The specific implementation is described in Thompson *et al.* (1996). The implementation achieves second-order time accuracy and spectral spatial convergence as the number of nodes per element is increased. The software has been successfully used on a number of related problems (*e.g.*, flow past plates, Thompson *et al.* 1996), and three-dimensional circular cylinder wake transition (Thompson *et al.* 1996). Resolution and domain size studies indicate the accuracy of the predictions is better than 1%. The critical Reynolds number is $Re = 46.4$ for the present simulations. The exact transition Reynolds number is slightly dependent on domain blockage; Dusek *et al.* (1994) found a critical value of $Re = 46.1$.

3.2. SIMULATIONS AT SLIGHTLY POST-CRITICAL REYNOLDS NUMBERS

Simulations were performed at a number of Reynolds numbers exceeding the critical Reynolds number, however, for the purpose of the present discussion we will focus on the $Re = 48$ case, which is representative of the general behaviour. The growth and saturation of the transverse velocity component at $2R$ downstream is shown in figure 1(a). The transition takes place naturally, from initiation through computer round-off error, without the need to add a random noise component. At this Reynolds number the shedding period is 16.48 R/U_∞, hence it takes approximately two-hundred periods to grow from low levels to saturation. Note that the timestep was 0.01 R/U_∞, corresponding to 1648 timesteps per shedding cycle. For the analysis described below, the transverse velocity was recorded at the following positions on the wake centerline: $\zeta = x/R =$ 1.3, 2, 4, 7, 10.5, 14, 17.5, 21, 24.5, 28, 31.5, 35, 38.5, 42, 45.5, 49, 52.5, 56.

Because of the distinct difference in timescale between the shedding period and the instability growth timescale, these signals can be accurately analysed to evaluate the Landau model coefficients. To do this, the times and amplitudes of all local peaks and troughs are extracted using quadratic interpolation. This provides direct measures of $\rho(t)$ and $\phi(t)$. Next the derivatives on the lefthand sides of equations (3) and (4), $d \log \rho/dt$ and $\dot{\phi}$, are evaluated by central differences. This provides two sets of derivatives as a function of ρ^2 and time. From these data, a fourth-order least-squares fit is performed. The functional variation is not strong, and a fourth-order fit captures the variation accurately.

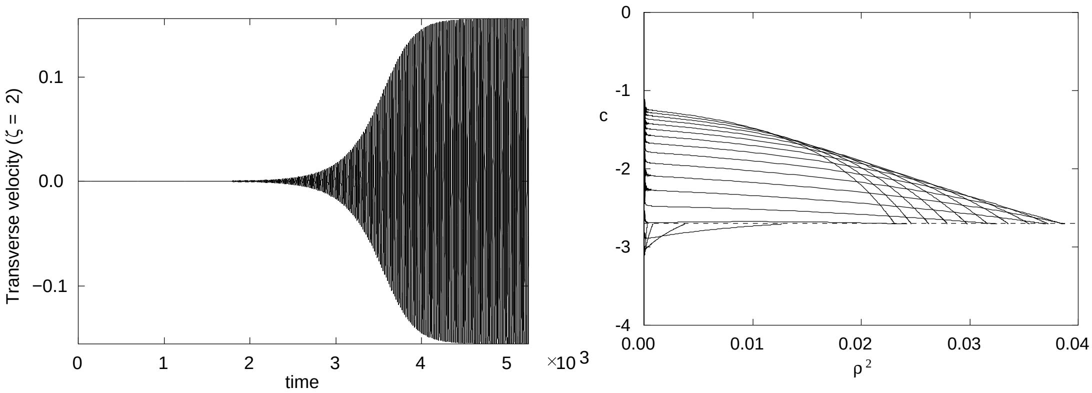

Figure 1. Left: Growth and saturation of the transition as depicted by the transverse velocity component at $\zeta = 2$ and $Re = 48$. Right: Variation of the *Landau constant* with ρ^2 and position. The different curves correspond to different downstream positions on the wake centreline as given in the text. The curves end on the dashed line; the y-coordinate corresponds to c_{sat}. The curves are ordered with the position closest to the cylinder having the most negative initial c.

The data can be further analysed to evaluate the Landau constant. This can be determined by plotting $c(\rho^2) = (d \log \rho/dt - \gamma)/(d\phi/dt - \omega)$ against ρ^2 for each point on the wake centreline. Here γ and ω are derived from a least-squares fit at each point. Figure 1(b) shows this variation. Clearly, this plot shows that the *Landau constant* indeed approaches a position-independent value as the flow saturates. When the flow is growing exponentially in the linear regime, the Landau constant is a strong function of position, varying between approximately -3 close to the back of the cylinder to approximately -1 far downstream. At saturation $c_{sat} = -2.708$. This is consistent with the value found by Dusek *et al.* (1994). However, these authors found the Landau constant was only close to a constant independent of position and amplitude. The difference between the result here and the previous finding results from the different ways the Landau constant was evaluated. Dusek *et al.* (1994) evaluated the Landau constant by performing a linear least-squares fit to estimate c_R and c_I with a long time series of data including a considerable period of time after the flow had saturated. This leads to a biasing of the parameters to their saturated values. Hence their estimated Landau constants at different downstream positions were close to the saturated position-independent value found here. Numerically they found a variation of about 3%. In contrast, from the current numerical results, it is found that the Landau constant at saturation is position-independent to within 0.05% over the range ($1.3R < x < 56R$). This is within numerical error associated with the finite-differencing. Of interest, at approximately $10R$ downstream the Landau constant remains approximately constant during linear growth and saturation.

4. DISCUSSION AND CONCLUSIONS

Accurate numerical simulations have been used to analyse the applicability of the Stuart-Landau model to the initial Hopf bifurcation of the wake of the circular cylinder. While previous numerical and experimental results have indicated that the model appears to work remarkably well, a closer examination reveals that the story is more complex. For example, Dusek *et al.* (1994) found that, if the Landau model variable is taken as the transverse velocity on the centreline, the Landau *constant* is position independent to within about 3% over a range of different positions in the wake. We find that the value they found corresponds to what we have called the Landau constant at saturation (c_{sat}). Here, we focus on the truncated cubic Stuart-Landau model and allow higher-order terms to be accounted for by allowing the complex cubic coefficient to be a function of amplitude. In this case, it can be shown that given a few physically realistic assumptions (*i.e.*, phase-locking during the linear growth phases and after saturation), c_{sat} must become a constant independent of position (although still dependent on Reynolds number). The numerical results bear this out to within numerical error. On the other hand, the initial Landau constant (c_0), *i.e.*, the value straight after the linear growth phase when cubic terms start to be important, is far from constant and is found to vary with position by a factor of approximately 3. Importantly, this parameter is really the mathematical Landau constant corresponding to the semi-infinite Stuart-Landau model containing odd terms in the amplitude that has been treated previously analytically.

These results apply to a local wake variable (the transverse velocity) rather than a global variable (as perhaps the model was suggested for). To investigate the applicability of these results to a global variable, the Stuart-Landau model was examined also using the lift coefficient per unit length of the cylinder as the Landau model variable. In so far as the local results can be carried across, they apply to the analysis using the lift coefficient. For example, post-saturation c_{sat} is the same constant as for the local analysis. There is also non-negligible variation of the Landau constant with amplitude during the saturation phase as there is for the transverse velocity. The model will be further explored at the conference.

References

1. ALBAREDE, P. & PROVANSAL, M. 1995 Quasi-periodic cylinder wakes and the Ginzburg-Landau model. *J. Fluid Mech.* **291**, pp. 191–222.
2. BARKLEY, D., TUCKERMANN, L.S. and GOLUBITSKY, M. 2000 Bifurcation Theory for Three-Dimensional Flow in the Wake of a Circular Cylinder *Phys. Rev. E* **61**, 5, 5247–5252.
3. DUSEK, J., LE GAL, P. and FRAUNIE, P. 1994 A numerical and theoretical study of the first Hopf bifurcation in a cylinder wake, *J. Fluid Mech.* **264**, 59–80.
4. GAMBAUDO, J.M. 1985 Perturbation of a Hopf bifurcation by an external time-periodic forcing. *J. Differential Equations* **57**, pp. 172–199.
5. GHIDERSA, B. and DUSEK. J. 2000 Breaking of axisymmetry and onset of unsteadiness in the wake of a sphere *J. Fluid Mech.* **423**, 33–69.
6. HENDERSON, R.D. 1997, Nonlinear Dynamics and Pattern Formation in Turbulent Wake Transition, *J. Fluid Mech.*, **352**, 65.
7. LE GAL, P., NADIM, A. and THOMPSON, M.C. 2001 Hysteresis in the Forced Stuart-Landau Equation: Application to Vortex Shedding From an Oscillating Cylinder *J. Fluids Struct.* **15**, 3/4, 445–457.
8. MARGARVEY, R.H. & BISHOP, R.L., 1961 Wakes in liquid-liquid systems. *Phys. Fluids* **4** (7) 800–805.
9. ORMIÈRES, D. & PROVANSAL, M. 1999 Transition to turbulence in the wake of a sphere. *Phys. Rev. Lett.* **83**, 80–83.
10. PESCHARD, I. 1995 De L'Oscillateur aux Sillages Couplés. *Ph.D. Thesis*, Institut de Recherches sur les Phénomènes Hors Equilibre, Marseille, France.
11. PROVANSAL, M. & ORMIÈRES, D. 1998 Etude expérimentale de l'instabilité du sillage d'une sphère. *C. R. Acad. Sci. Paris* **326** (Série II b), 489–494.
12. PROVANSAL, M., MATHIS, C. & BOYER, L. 1987 Bénard-von Karman instability: transient and forced regimes. *J. Fluid Mech.* **182**, pp. 1–22.
13. SCHUMM, M., BERGER, E. & MONKEWITZ P. 1994 Self-excited oscillations in the wake of two-dimensional bluff bodies and their control. *J. Fluid Mech.* **271**, pp. 17–53.
14. SREENIVASAN, K.R., STRYKOWSKI, P.J. & OLINGER D.J. 1986 Hopf bifurcation, Landau equation, and vortex shedding behind circular cylinders. In *Proc. Forum on Unsteady Flow Separation, ed. K. N. Ghia.* ASME FED **52**, pp. 1–13.
15. THOMPSON, M.C., HOURIGAN, K. & SHERIDAN, J. 1996 Three-Dimensional instabilities in the wake of a circular cylinder. *Exp. Therm. Fluid Sci.* **12**, 190.
16. THOMPSON, M.C., LEWEKE, Th. and PROVANSAL, M. 2001 Kinematics and Dynamics of Sphere Wake Transition *J. Fluids Struct.* **15**, 3/4, 575–585.
17. WILLIAMSON, C.H.K. & ROSHKO, A. 1988 Vortex formation in the wake of an oscillating cylinder. *J. Fluids Struct.* **2**, pp. 355–381.
18. WILLIAMSON, C.H.K. 1988, The existence of two stages in the transition to three-dimensionality of a cylinder wake, *Phys. Fluids*, **31**, 3165.
19. ZIELINSKA, B.J.A., WESFREID, J.E. 1995 On the Spatial Structure of Global Modes in Wake Flows *Phys. Fluids* **7**, 6, 1418–1424.