Assumed Stress-Strain Four-Node Elements Based on the Finite Deformation First-Order Theory of Multilayered Shells

G. M. Kulikov[1], S. V. Plotnikova[1]

Summary

This paper presents assumed stress-strain four-node curved shell elements with six displacement degrees of freedom per node for the finite deformation first-order shell theory. The developed formulation is based on the principally new non-linear strain-displacement relationships that are objective, i.e., invariant under arbitrarily large rigid-body motions. To avoid shear and membrane locking and have no spurious zero energy modes, the assumed stress resultant and strain fields are invoked. In order to circumvent thickness locking, the modified material stiffness matrices corresponding to the plane stress state are employed. The fundamental unknowns consist of six displacements and 11 strains of the face surfaces of the shell, and 11 stress resultants. The element characteristic arrays are obtained by using the Hu-Washizu variational principle. To demonstrate the efficiency and accuracy of this formulation and to compare its performance with other non-linear finite element models reported in the literature, two numerical examples are presented.

Problem Formulation

Using the solid-shell concept in a non-linear finite element (FE) formulation is well established and has been shown to give acceptable results [1-4]. In order to develop the solid-shell elements that overcome shear, membrane, trapezoidal and thickness locking, advanced FE techniques were applied. In this light, in some works for constructing the solid-shell elements only displacements of the face surfaces are used. A main idea of such approach is that displacement vectors of the face surfaces of the shell are represented in some global Cartesian basis in order to exactly describe rigid-body motions.

Herein, it is developed a close non-linear FE formulation based on the first-order theory of multilayered beams [5], plates [6] and shells [7]. But in our FE development selecting as unknowns the displacements of the face surfaces in a form

$$\mathbf{u} = N^{-}(\alpha_3)\mathbf{v}^{-} + N^{+}(\alpha_3)\mathbf{v}^{+}, \tag{1a}$$

$$\mathbf{u} = \sum_i u_i \mathbf{e}_i, \qquad \mathbf{v}^{\pm} = \sum_i v_i^{\pm} \mathbf{e}_i \tag{1b}$$

[1] Tambov State Technical University, Sovetskaya Street, 106, Tambov 392000, Russia

has a principally another mechanical sense and allows to formulate curved shell elements with very attractive properties, since objective non-linear strain-displacement relationships [7], i.e., invariant under all large rigid-body motions are applied. In approximation (1) the following notations are used: $\mathbf{u}$ is the displacement vector; $u_i(\alpha_1,\alpha_2,\alpha_3)$ are the components of this vector, which are always measured in accordance with the total Lagrangian formulation from the initial configuration to the current configuration directly; $\mathbf{v}^{\pm}$ are the displacement vectors of face surfaces $S^{\pm}$; $v_i^{\pm}(\alpha_1,\alpha_2)$ are the components of these vectors; α_1 and α_2 are the orthogonal curvilinear coordinates of the reference surface; α_3 is the normal coordinate; $\mathbf{e}_1$ and $\mathbf{e}_2$ are the tangent unit vectors to the lines of principal curvatures; $\mathbf{e}_3$ is the vector normal to the reference surface; $N^{\pm}(\alpha_3)$ are the linear shape functions of the shell.

Taking into account that displacement vectors of the face surfaces (1b) are represented in the *local* reference surface basis, the developed FE formulation has computational advantages compared to conventional isoparametric FE formulations because it eliminates the costly numerical integration by deriving the stiffness matrix. Besides, our element matrix requires only direct substitutions, i.e., no inversion is needed if sides of the element coincide with the lines of principal curvatures of the reference surface, and it is evaluated by using the full exact *analytical* integration.

The FE formulation is free of assumptions of small displacements, small rotations and small loading steps because it is based on the objective fully non-linear strain-displacement relationships. There exists only one limitation associated with a simple fact that a loading step cannot be too large. This restriction arises in a case of using the Newton-Raphson method for solving equilibrium equations for incremental nodal degrees of freedom, i.e., in work [7] the incremental assumed strains and stress resultants are eliminated at the element level. Herein, it is discussed an alternative approach when equilibrium equations for incremental displacements and incremental assumed strains and stress resultants are solved by the Newton-Raphson method simultaneously. As a result an additional incremental load vector due to compatibility mismatch [3, 4, 8] is present and disappears only at the end of the iteration process. So, this refined approach allows to use much larger load increments in comparison with approach [7].

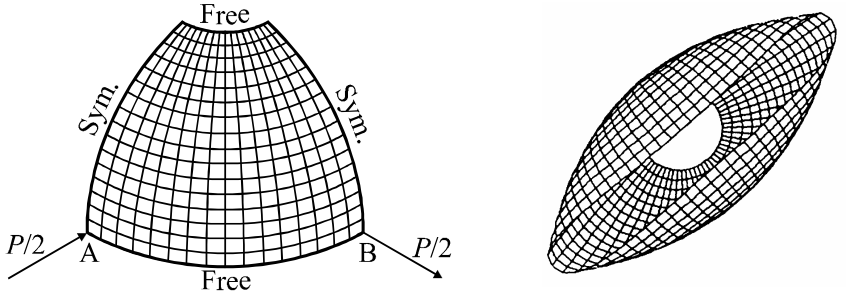
The proposed FE formulation is based on a simple and efficient approximation of shells via four-node *curved* elements. To avoid shear and membrane locking and have no spurious zero energy modes, the assumed stress resultant and strain fields are invoked. In order to circumvent thickness locking, the modified material stiffness matrices symmetric [1, 3, 9] or non-symmetric [5-7] corresponding to the plane stress state are employed. As a result, two elements were constructed, namely, TMS4SA with a symmetric stiffness matrix and TMS4RA

with a non-symmetric one. Note also that fundamental unknowns consist of six displacements and 11 strains of the face surfaces of the shell, and 11 stress resultants. Therefore, for deriving element characteristic arrays the Hu-Washizu variational principle should be applied.

Numerical Tests

In all numerical benchmark problems the tolerance error from the standard criterion [7] is set to be $\varepsilon = 10^{-4}$. All our results are compared with those based on using identical node spacing and in the first example the same convergence tolerance. Besides, NStep denotes a number of load steps employed to equally divide the maximum load while NIter stands for a number of iterations. Note also that the developed shell elements TMS4RA and TMS4SA are slightly distinguishable for engineering calculations, and the predictions of all elements are insensitive to a number of loading steps.

To investigate the capability of TMS4RA and TMS4SA elements to overcome membrane and shear locking phenomena, we consider one of the most demanding non-linear test. A hemispherical shell with 18^o hole at the top is loaded by two pairs of opposite concentrated forces on the equator. The geometrical and material data of the problem are shown in Fig.1. Due to symmetry of the problem, only one quarter of the shell is modeled with 16×16 mesh of the TMS4 ele-



$R = 10,\ h = 0.04,\ \text{Hole} = 18^o,\ E = 6.825 \times 10^7,\ \nu = 0.3,\ P = 100f,\ f = 4$

Shell of revolution with geometrical parameters[2]:

$A_1 = R,\ A_2 = R\cos\varphi,\ k_1 = k_2 = 1/R,\ \alpha_1 = \varphi \in [0,\ 2\pi/5],\ \alpha_2 \in [0,\ \pi/2]$

Fig. 1. Pinched hemispherical shell

[2] A_1 and A_2 are the Lamé coefficients; k_1 and k_2 are the principal curvatures of the reference surface

ments. Table 1 and Fig. 2 present a comparison with solutions [3, 4] derived by using 8×8 and 16×16 meshes of biquadratic and bilinear solid-shell elements, correspondingly. As can be seen, both TMS4 elements perform very well, since only 7 iterations are needed to obtain a solution of this discriminating problem.

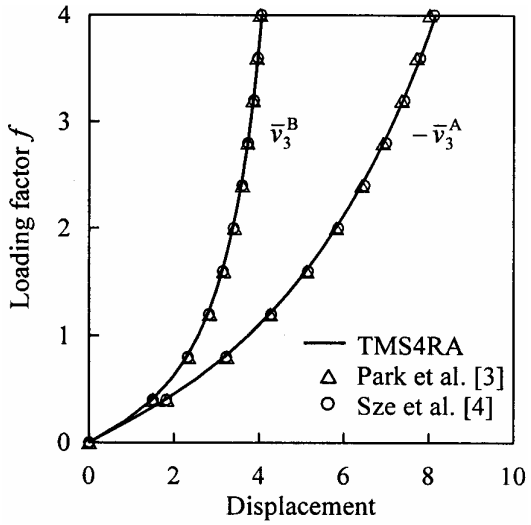


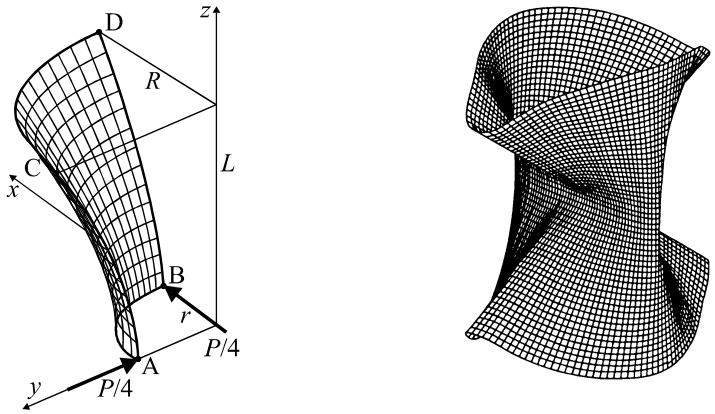
Fig. 2. Displacements of pinched hemispherical shell

Table 1. Transverse displacements under applied loads of hemispherical shell

Element	NStep = 1			NStep = 5			NStep = 10		
	$\bar{v}_3^B$	$-\bar{v}_3^A$	NIter	$\bar{v}_3^B$	$-\bar{v}_3^A$	NIter	$\bar{v}_3^B$	$-\bar{v}_3^A$	NIter
TMS4RA	4.0566	8.1519	7	4.0566	8.1519	17	4.0566	8.1519	29
TMS4SA	4.0568	8.1523	7	4.0568	8.1523	17	4.0568	8.1523	29
Park et al.	4.0205	8.0160	8	4.0209	8.0169	23	4.0209	8.0169	35
Sze et al.	4.0488	8.1173	8	4.0488	8.1173	21	4.0488	8.1173	36

Further we consider a cross-ply hyperbolic shell under two pairs of opposite concentrated forces. The geometrical and material data of the three-layer hyperbolic shell are given in Fig. 3, where 0^o and 90^o refer to the circumferential and meridional directions. Owing to symmetry of the problem, only one octant of the shell is discretized with the uniform 28×28 mesh of TMS4 elements. Table 2 and Fig. 4 present our results compared with those reported in works [2, 10] using 14×14 and 28×28 uniform meshes of biquadratic and bilinear solid-shell

elements, where $\overline{u}_x$ and $\overline{u}_y$ denote displacements of the middle surface in x and y directions. One may observe that both TMS4 elements perform excellently.



$$r = 7.5,\ R = 15,\ L = 20,\ h = 0.04,\ E_1 = 4\times10^7,\ E_2 = E_3 = 10^6$$

$$G_{12} = G_{13} = G_{23} = 6\times10^5,\ \nu_{12} = \nu_{13} = \nu_{23} = 0.25,\ P = 40f,\ f = 10$$

$$\text{Ply thickness} = h/3,\ \text{Ply orientation} = [90/0/90]$$

Shell of revolution with geometrical parameters $(\alpha_1 = z \in [0,\ L],\ \alpha_2 \in [0,\ \pi/2])$:

$$A_1 = \sqrt{1+\frac{\mu^2 z^2}{A_2^2}},\quad A_2 = r\sqrt{1+\frac{\mu z^2}{r^2}},\quad k_1 = -\frac{\mu r^2}{A_1^3 A_2^3},\quad k_2 = \frac{1}{A_1 A_2},\quad \mu = \frac{R^2 - r^2}{L^2}$$

Fig. 3. Pinched cross-ply hyperbolic shell

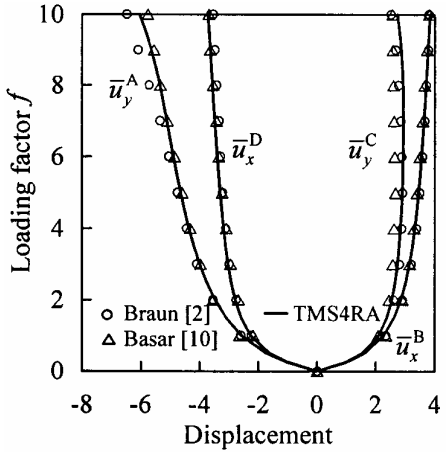


Fig. 4. Displacements of pinched cross-ply hyperbolic shell

Table 2. Displacements at points A, B, C and D of cross-ply hyperbolic shell

Element	NStep =4					NStep =10				
	$-\overline{u}_y^{\mathrm{A}}$	$\overline{u}_x^{\mathrm{B}}$	$\overline{u}_y^{\mathrm{C}}$	$-\overline{u}_x^{\mathrm{D}}$	NIter	$-\overline{u}_y^{\mathrm{A}}$	$\overline{u}_x^{\mathrm{B}}$	$\overline{u}_y^{\mathrm{C}}$	$-\overline{u}_x^{\mathrm{D}}$	NIter
TMS4RA	6.0663	3.8311	2.7258	3.7339	25	6.0663	3.8311	2.7258	3.7339	36
TMS4SA	6.0660	3.8310	2.7257	3.7340	25	6.0660	3.8310	2.7257	3.7340	36

Reference

1 Kim, Y.H. and Lee, S.W. (1988): "A solid element formulation for large deflection analysis of composite shell structures", *Comp. Struct*., Vol. 30, pp. 269-274.

2 Braun, M., Bischoff, M. and Ramm, E. (1994): "Nonlinear shell formulations for complete three-dimensional constitutive laws including composites and laminates", *Comput. Mech*., Vol. 15, pp. 1-18.

3 Park, H.C., Cho, C. and Lee, S.W. (1995): "An efficient assumed strain element model with six dof per node for geometrically nonlinear shells", *Int. J. Numer. Meth. Eng*., Vol. 38, pp. 4101-4122.

4 Sze, K.Y., Chan, W.K. and Pian, T.H.H. (2002): "An eight-node hybrid-stress solid-shell element for geometric non-linear analysis of elastic shells", *Int. J. Numer. Meth. Eng*., Vol. 55, pp. 853-878.

5 Kulikov, G.M. and Plotnikova, S.V. (2004): "Non-conventional non-linear two-node hybrid stress-strain curved beam elements", *Finite Elements Anal. Des.*, 27 (journal pages).

6 Kulikov, G.M. and Plotnikova, S.V. (2004): "Finite deformation plate theory and large rigid-body motions", *Int. J. Non-Linear Mech.*, Vol. 39, pp. 1093-1109.

7 Kulikov, G.M. and Plotnikova, S.V. (2003): "Non-linear strain-displacement equations exactly representing large rigid-body motions. Part I. Timoshenko-Mindlin shell theory", *Comput. Meth. Appl. Mech. Eng*., Vol. 192, pp. 851-875.

8 Atluri, S. (1973): "On the hybrid stress finite element model for incremental analysis of large deflection problems", *Int. J. Solids Struct*., Vol. 9, pp. 1177-1191.

9 Kulikov, G.M. and Plotnikova, S.V. (2002): "Simple and effective elements based upon Timoshenko-Mindlin shell theory", *Comput. Meth. Appl. Mech. Eng*., Vol. 191, pp. 1173-1187.

10 Basar, Y., Itskov, M. and Eckstein, A. (2000): "Composite laminates: nonlinear interlaminar stress analysis by multi-layer shell elements", *Comput. Meth. Appl. Mech. Eng*., Vol. 185, pp. 367-397.