A new stable discontinuous Galerkin approximation for non-linear conservation laws on adaptively refined grids

Andreas Dedner[a], Charalambos Makridakis[b], Mario Ohlberger[a,*]

[a] *Abteilung für Angewandte Mathematik, Universität Freiburg, Hermann-Herder-Str.10, D-79104 Freiburg, Germany*
[b] *Department of Applied Mathematics, University of Crete, GR-71409 Heraklion, Greece*

Abstract

We propose an a-posteriori error estimate for the semi-discrete discontinuous Galerkin (DG) method of arbitrary order in arbitrary space dimensions. For stabilization of the scheme a general framework of projections is introduced. Finally it is demonstrated numerically how the a posteriori error estimate is used for defining appropriate projection operators and in order to design an efficient grid adaption strategy. Numerical experiments show the gain in efficiency in comparison with computations on uniform grids.

Keywords: Discontinuous Galerkin; Higher order; Adaptive methods; Error estimate; Finite element; Conservation laws

1. Introduction

In this contribution we study a semi-discrete version of the discontinuous Galerkin (DG) approximation of Cockburn and Shu [1] for non-linear scalar conservation laws in several space dimensions. As a prototype conservation law, consider the Cauchy initial value problem:

$$\partial_t u + \nabla \cdot f(u) = 0 \quad \text{in } \mathbb{R}^d \times \mathbb{R}^+ \tag{1a}$$

$$u(x,0) = u_0(x) \text{ in } \mathbb{R}^d. \tag{1b}$$

Here $u(x, t)$: $\mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ denotes the dependent solution variable, $f(u) \in C^1(\mathbb{R})$ denotes the flux function, and $u_0(x)$: $\mathbb{R}^d \to \mathbb{R}$ the initial data.

It is well known (see [2] and the references therein) that (1a) and (1b) admit a unique entropy weak solution in the class of functions of bounded variation (BV).

In this contribution we give an a posteriori error estimate to a generalized semi-discrete class of discontinuous Galerkin approximations of (1a) and (1b). Using this error estimate we define a new class of limiting projections that are used to stabilize the scheme. In addition, we use the a posteriori result for local adaptive mesh refinement.

The paper is organized as follows. In Section 2 we will introduce some notation and define the class of semi-discrete DG approximations. In Section 3 we will state the a posteriori error estimate. A special choice within the new class of DG approximations is specified in Section 4, and finally in Section 5 we demonstrate the good convergence behaviour of our new approach in a numerical experiment.

2. Notation and generalized formulation of the DG method

Let $\mathcal{T}$ denote a tessellation of $\mathbb{R}^d$ with control volumes $T \in \mathcal{T}$ such that $\cup_{T\in\mathcal{T}} \bar{T} = \mathbb{R}^d$. Let h_T denote a length scale associated with each control volume T, e.g. $h_T \equiv$ diam (T). For two distinct control volumes T_i and T_j in $\mathcal{T}$, the intersection is either an oriented edge (2-D) or face (3-D) S_{ij} with oriented normal ν_{ij} or else a set of measure at most $d-2$. The set of edges or face of the tessellation $\mathcal{T}$ will be denoted by Γ.

On the tessellation $\mathcal{T}$ we define the discontinuous space of piecewise polynomials of degree p by $V_h^p := \{v_h \in L^1(\mathbb{R}^d) \, | v_h|_T \in \mathbb{P}_p \text{ for all } T \in \mathcal{T}\}$. Let us denote by $\prod_{V_h^p}$ the L^2-projection into V_h^p. Furthermore, $[v_h]|S_{ij} := (v_j|S_{ij}^p - v_l \mid S_{ij})\, \nu_{ij}$ is the jump of v_h on the edge S_{ij}, and $\{v_h\}\, S_{ij} := 1/2\ (v_j|_{S_{ij}} + v_l|_{S_{ij}})$ denotes the mean of v_h at an

* Corresponding author. Tel.: +49 76 12035642; Fax: +49 76 12035632; E-mail: mario@mathematik.uni-freibrug.de

Computational Fluid and Solid Mechanics 2005
K.J. Bathe (Editor)

interface. With these notations we are ready to define the semi-discrete DG scheme.

Definition 1 (semi-discrete DG approximation). $u_h : C^1(0, T; V_h^p)$ is called a semi-discrete DG approximation of (1a) and (1b), if

$$u_h(0) = \prod_{V_h^p}(u_0) \tag{2a}$$

$$\frac{d}{dt}(u_h(t), v_h) - (f(u_h(t)), \nabla v_h) + (f_h(u_h(t)), [v_h])_\Gamma = 0$$
$$\text{for all } v_h \in V_h^p \tag{2b}$$

Here $(\cdot,\cdot)$ denotes the L^2 inner product, $(\cdot,\cdot])r$ denotes the L^2 inner product on the set of inter, face Γ, and f_h denotes a numerical flux function that is uniquely defined on the interfaces of the tessellation. Detailed assumptions on f_h will be stated below.

There is numerical evidence and it is well known in literature that the DG method is not stable for a higher polynomial degree. Therefore, in the next step we are going to introduce limiting projections in the discretization that will be chosen in Section 4. We will introduce the projections in the semi-discrete formulation, which enforces us to introduce a partition of the time interval $(0, T)$. Thus, let $\{0 = t^o, \ldots, t^N = T\}$ denote a partition of $(0, T)$ and let us define the time increment $\Delta t^n := t^{n+1} - t^n$. In addition we define a local projection operator. Let us define $\bar{v}_h$ through $\bar{v}_j := \prod_{V_h^p}(v)|_{T_j}$ for any $v \in L^2(\Omega)$. Furthermore, let us introduce a mapping $\Lambda_h : [0,T] \times V_h^p \to V_h^p$, such that $\Lambda_h^{n,t} := \Lambda_h(t, \cdot)$ are projections on V_h^p for all $t \in [t^n, t^{n+1}]$ with the properties that for any $v_h \in H^1(0, T; V_h^p)$ we have

$$\overline{v_h(t)} = \overline{\Lambda_h^{n,t}(v_h(\cdot, t))} \tag{3}$$

and $\Lambda_h^{n,\cdot}(v_h(\cdot,\cdot)) \in C^1((t^n, t^{n+1}); V_h^p$. Note that $\Lambda_h(t, v_h(\cdot, t))$ may, in general, have discontinuities in time at the points $t = t^n, n = 1, 2, \ldots$ We define the restrictions $\Lambda_j^{n,t}: V_h^p \to \mathbb{P}_p(T_j)$ by

$$\Lambda_j^{n,t}(v_h(\cdot, t))(x) := \Lambda_h^{n,t}(v_h(\cdot, t))(x),$$
$$\text{for all } (x,t) \in T_j \times [0, T]$$

We now define the generalized semi-discrete DG approximation.

Definition 2 (Generalized semi-discrete DG approximation). For all $n = 0, \ldots, N$ and $T_j \in \mathcal{T}_h$ let us suppose that a projection $\Lambda_h^{n,t} : V_h^p \to V_h^p$ with property (3) is given $u_h \in H^1(0, T; V_h^p)$ is called a generalized semi-discrete DG approximation of (1a) – (1b), if for $u_h^{-1} := \Lambda_h^{0,0}(u_0)$ we have:

For $n = 0, \ldots, N-1$, $u_h^n := u_h|_{[t^n, t^{n+1})} \in C^1(t^n, t^{n+1}; V_h^p)$ is defined through

$$u_h^n(t^n) := \Lambda_h^{n,t^n}(u_h^{n-1}(t^n)), \tag{4a}$$

$$\frac{d}{dt}(u_j^n(t), v_j)_{T_j} = -\sum_{l \in N(j)}(f_{jl}(\Lambda_j^{n,t}(u_h^n(t)), \Lambda_l^{n,t}(u_h^n(t))), v_j)s_{jl} + (f(\Lambda_j^{n,t}(u_h^n(t))), \nabla v_j)_{T_j}, \tag{4b}$$

for all $v_j \in \mathbb{P}_p, j \in I, t \in [t^n, t^{n+1}]$.

Here $N(j)$ denotes the index set of neighboring control volumes to T_j, $(\cdot,\cdot)T_j$, $(\cdot,\cdot)S_{jt}$ denote the local inner product on T_j, S_{jl} respectively, and $f_{jl}(u_j(t), u_l(t))$ is the restriction of $f_h(u_h)$ to S_{jl}.

3. A-posteriori error estimate

Theorem 3.1 (A-posteriori error estimate for the semi-discrete DG method). Let u be the unique entropy solution of (1a)–(1b) and let u_h be given by the semi-discrete generalized DG method (4a)–(4b). Let us denote $\tilde{u}_h(t) := \Lambda_h^t(\cdot, t)$) Then the following a-posteriori error estimate holds for given constants K_1, K_2

$$\|(u - \tilde{u}_h)(T)\|_{L^1(B_R(x_0))} \leq \eta_h$$

where $\eta_h := \eta_0 + \sqrt{K_1\eta_1} + \sqrt{K_2\eta_2}$, $\eta_0 := \sum_n \sum_{j \in I_t} \eta_{i,j}^n$, i = 1, 2, and the local contributions $\eta_{i,j}^n$ are given as $\eta_{0,j} := \int_{T_j} |u_0 - \tilde{u}_j^0(0)|$,

$$\eta_{1,j}^n := \int_{t^n}^{t^{n+1}} \int_{T_j} h_j |\partial_t \tilde{u}_j + \nabla \cdot f(\tilde{u}_j)| + \frac{1}{2}\int_{t^n}^{t^{n+1}} h_{jl} \int_{S_{jl}} Q_{jl}(\tilde{u}_j, \tilde{u}_l)|\tilde{u}_j - \tilde{u}_l| + \int_{T_j} h_j |\tilde{u}_j^{n+1}(t^{n+1}) - \tilde{u}_j^n(t^{n+1})| \tag{5a}$$

$$\eta_{2,j}^n := \int_{t^n}^{t^{n+1}} \left\|\overline{\tilde{u}_j^n} - \tilde{u}_j^n\right\|_{L^\infty(T_j)} \int_{T_j} \left|\partial_t \tilde{u}_j^n + \nabla \cdot f(\tilde{u}_j^n)\right| + \frac{1}{2}\int_{t^n}^{t^{n+1}} \max_{k \in \{j,l\}} \left\|\overline{\tilde{u}_k^n} - \tilde{u}_k^n\right\|_{L^\infty(S_{jl})} \int_{S_{jl}} Q_{jl}(\tilde{u}_j, \tilde{u}_l)|\tilde{u}_j, \tilde{u}_l| + \left\|\overline{\tilde{u}_j^n}(t^{n+1}) - \tilde{u}_j^n(t^{n+1})\right\|_{L^\infty(T_j)} \int_{T_j} \left|\tilde{u}_j^{n+1}(t^{n+1}) - \tilde{u}_j^n(t^n)\right| \tag{5b}$$

Here, we used the notation

$$Q_{jl}(u, v) := \frac{2f_{jl}(u, v) - f_{jl}(u, u) - f_{jl}(v, v)}{u - v},$$
$$h_{jl} := diam(T_j \cup T_l)$$

Proof. The proof of this theorem is given in [3].

4. A specific choice of the projection operators

For the choice of the projection operators $\Lambda_h^{n,t}$ in the generalized DG method 2.2 we are now going to introduce our new approach that is based on a restriction of the gradient of the approximate solution based on the error estimate in Theorem 3.1. Together with a local mesh adaption strategy this method is then used in an *hp*-adaptive manner.

The goal of the choice of the projection $\Lambda n, t h$ is twofold. On the one hand we need a projection or limiting of the solution in order to stabilize the scheme at least in the case of non-linear conservation laws and, on the other hand, we would like to bound the term $\|\overline{\tilde{u}_j^n(\cdot,t)} - \tilde{u}_j^n(\cdot,t)\|_{L\infty(Tj)}$ in the error indicator $\eta_{2,j}^n$ of Theorem 3.1 in order to get a converging error indicator for successively refined computational meshes.

A comparison of the indicators $\eta_{2,j}^n$ and $\eta_{3,j}^n$ shows that the projection should be done in such a way that $\|\overline{\tilde{u}_j^n(\cdot,t)} - \tilde{u}_j^n(\cdot,t)\|_{L\infty(Tj)} = \mathcal{O}(h_j)$. On the other hand the projection should only be active on mesh cells near discontinuities and not in smooth regions with steep gradients. Thus, we suggest to define a projection parameter λ_h as

$$\lambda_j^n(t) := \frac{h_j}{(h_j + r_j^n)^{\frac{p+2}{p+1}}}, \quad r_j^n := \int_{T_j} \left|\partial_t \tilde{u}_j + \nabla \cdot f(\tilde{u}_j)\right| + \frac{1}{2} \int_{S_{jl}} Q_{jl}(\tilde{u}_j, \tilde{u}_l) \left|\tilde{u}_j - \tilde{u}_l\right| \tag{6}$$

and to ensure that our projection operators yields a solution with the property

$$\left\|\overline{\tilde{u}_j^n(\cdot,t)} - \tilde{u}_j^n(\cdot,t)\right\|_{L^\infty(T_j)} \leq \lambda_j^n(t) \tag{7}$$

where we use the notation $\tilde{u}_j^n(\cdot,t) := \Lambda_j^{n,t}(u_h(\cdot,t))$.

In order to define the method, let φ_l, $l = 0, \ldots, p$ denote the orthogonal basis of Legendre polynomials on the cell $T_j := (x_{j+1/2}, x_{j-1/2})$ such that $\varphi_l \in \mathbb{P}_l(T_j)$. We then have the local expansion

$$u_j^n(x,t) = \sum_{l=0}^{p} u_{j,l}^n(t)\varphi_l(x) \tag{8}$$

where $\varphi_0 = 1$ and thus $\overline{u_j^n(\cdot,t)} = u_{j,0}^n(t)$

Definition 4.1 (Derivatives restriction method in 1D). Let $1 \leq l^* \leq p$ denote the maximal index such that

$$\sum_{l=1}^{l^*} u_{j,l}^n(t^n)\varphi_l(x) \leq \lambda_j^n(t^n), \quad \text{for all } x \in T_j$$

The projection operator in our new derivatives restriction method is then defined as

$$\Lambda_j^{n,t}(u_h(\cdot,t) \quad := \sum_{l=0}^{l^*} u_{j,l}^n(t)\varphi_l(x) + \tilde{u}_{j,l^*+1}^n(t)\varphi_{l^*+1}(x) \tag{9}$$

with $\tilde{u}_{j,l^*+1}^n(t) := sgn\,(u_{j,l^*+1}^n(t))$

$$\min\left\{\left|u_{j,l^*+1}^n(t)\right|, \lambda_j^n(t^n) - \left\|\sum_{l=0}^{l^*} u_{j,l}^n(t^n)\varphi_l\right\|_{L^\infty(T_j)}\right\}$$

5. Adaptive numerical experiments in one-space dimension

As a numerical example we look at the Buckley–Leverett equation which is a one-dimensional model for two-phase flow in porous media where capillary pressure effects are neglected. The unknown variable $u : (-1, 1) \times (0,0.4) \to \mathbb{R}$ is the saturation of the wetting phase within the two-phase mixture. It satisfies the non-linear conservation law

$$u_t + \partial_x f(u) = 0, \text{ on } (-1,1) \times (0,0.4),$$
$$u(\cdot,0) = u_0, \text{ on } (-1,1)$$

where the fractional flow rate f is given as $f(s) = \frac{u^2}{u^2 + \frac{1}{2}(1-u)^2}$. We will look at this problem for the following initial data.

$$u_0(x) := \begin{cases} 1, & \text{for } x < -0.6 \\ 0, & \text{for } -0.6 \leq x < 0.2 \\ 1, & \text{for } 0.2 \leq x \end{cases}$$

Thus, the solution of our Buckley–Leverett problem consists of the solution of two distinct Riemann problems for t smaller than some critical time T^*. The solution of each Riemann problem is a composed wave consisting of a rarefaction wave and an attached shock, and the exact solution is known up to solving an ODE for the rarefaction waves.

In Fig. 1 the adaptive DG-approximation with $p = 1$ is shown in comparison with the exact solution at $T = 0.4$. It can be seen that the adaptive DG approximation fits very well with the exact solution on a computational grid with only about 100 grid cells. The distribution of the refinement level of the local adaptive grid is also plotted. In Fig. 2 the convergence rate of our new DG scheme on uniform and on adaptive grids is shown. The a-posteriori error estimate of Theorem 3.1 is used for adaptive mesh refinement and for the definition of the projection operators (see Section 4). For a more detailed presentation of the adaptive strategy and for a comparison with other choices of limiting projection we refer to [3].

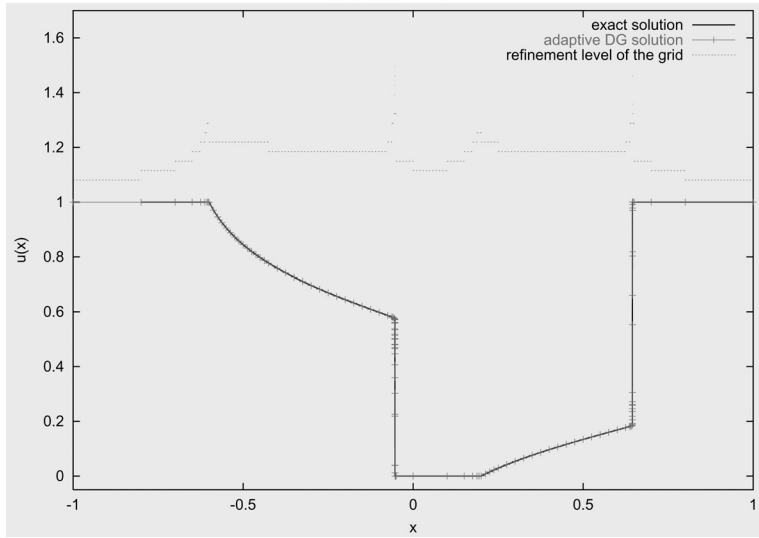

Fig. 1. Comparison of the adaptive DG approximation with the exact solution at $T = 0.4$ for an adaptive calulation with approximately 100 grid cells.

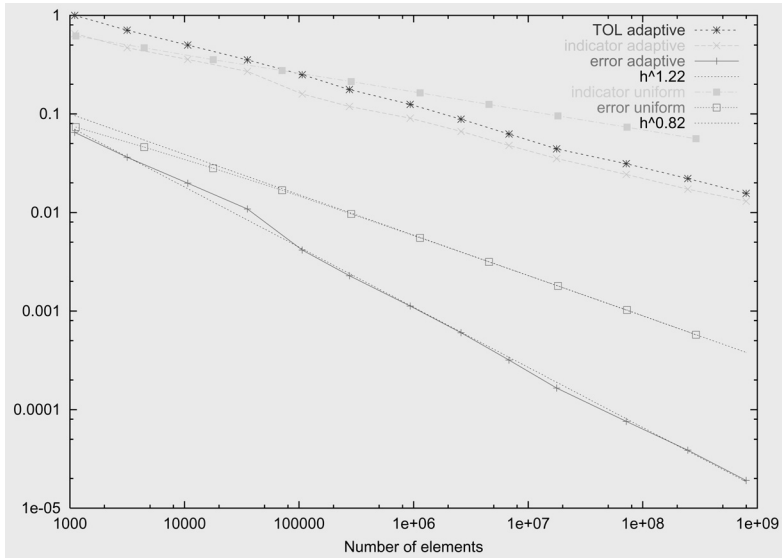

Fig. 2. Convergence study for the new DG scheme. The convergence rate on uniform refined grids is 0.82, while it is 1.22 on adaptively refined grids. In the adaptive case, the error indicator converges with the same rate as the prescribed tolerance TOL.

6. Conclusion

In this paper we introduced a new DG method for non-linear conservation laws that is derived from an a-posteriori error estimate. The resulting method uses limiting projections that are steered by the error indicators from our a posteriori result. In addition, the a-posteriori error estimate is also used for adaptive local mesh refinement. Numerical experiments demonstrate that the fully adaptive scheme has a better convergence order than a comparable scheme on uniform meshes.

References

[1] Cockburn B, Shu C-W. Runge–Kutta discontinuous Galerkin methods for convection-dominated problems. J Sci Comput 2001;16(3):173–261.

[2] Barth T, Ohlberger M. Finite volume methods: foundation and analysis. In: T.J.R. Hughes E. Stein, R. de Borst, editors, Encyclopedia of Computational Mechanics, Volume 1, Chapter 15. Chichester, UK: John Wiley, 2004.

[3] Dedner A, Makridakis C, Ohlberger M. Error control for a class of Runge-Kutta discontinuous Galerkin methods for nonlinear conservation laws. Preprint, Mathematisches Institut, Freiburg, 2005.