Breakdown revisited

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Abstract

The interplay between the axial and the azimuthal flow in the breakdown region near the axis of slender vortices is reinvestigated. The azimuthal momentum equation for axially symmetric flow is combined with the continuity equation to yield a simple differential equation for the angular velocity. It is found to be directly proportional to the axial velocity component along the axis for inviscid incompressible flow, and to the axial mass flux for compressible flow, exhibiting a behavior analogous to the variation of the cross-section area in subsonic and supersonic stream tubes. The Stokes stresses come into play in the immediate vicinity of the stagnation point.

Keywords: Slender vortices; Breakdown; Formation of stagnation point

1. Introduction

Longitudinal slender vortices have been investigated in numerous studies with the aim to predict the flow, in particular the process of the destruction of the core, usually referred to as vortex breakdown or bursting. Some of the references pertaining to this problem may be found in [1]. This continued interest is explained by the observation that slender vortices occur in a great variety of flows, for example in subsonic and supersonic flows along the leading edge of delta wings, in swirling pipe flows, in flows in rotating containers, in cascades of compressors at tip blades, and others. The flow structure in and outside of the vortex cores can strongly be affected when the pressure changes in the axial direction, either continuously or discontinuously: a pressure rise leads to a deceleration of the axial flow and a stagnation point may be formed. Also the azimuthal flow may be decelerated, and the vortex may break down.

In the present paper, the deceleration of inviscid and viscous axial flow in slender vortices is reinvestigated. Continuing the analysis of [1], the equations of motion for axially symmetric flow are simplified for the region near the axis of the vortex. The continuity equation and the azimuthal momentum equation can then be combined to yield a simple differential equation for the angular velocity. For inviscid flow a solution is readily obtained, exhibiting a strong influence of density variations on the velocity with different behavior for subsonic and supersonic axial flow. It is also found that the Stokes stresses cannot be neglected in the immediate vicinity of the stagnation point, when the axial velocity component tends to zero.

2. Solution of the governing equations for inviscid flow

2.1. Reduction of the governing equations

The flow in an inviscid, axially symmetric slender vortex with axial flow inside can be described by the Euler equations. If ρ denotes the density, p the pressure u, v, w the axial, radial, and the azimuthal velocity components, respectively, T the temperature, and x and r the axial and radial coordinates, the continuity equation and the azimuthal momentum equations can be written as

$$(\rho u)_x + (\rho v)_r + \rho v/r = 0 \tag{1}$$

$$uw_x + vw_r + vw/r = 0 \tag{2}$$

The subscripts indicate partial differentiation with respect to x and r. For $r \to O$ Eqs. (1) and (2) can be combined to yield

$$\Omega_x = \Omega(\rho u)_x/(\rho u) \tag{3}$$

where Ω denotes the angular velocity. Eq. (3) is used next to analyze the incompressible inviscid flow near the axis of vortex.

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2.2. Solution for incompressible inviscid flow

The differential equation for Ω, Eq. (3), can be integrated to yield

$$\Omega = \Omega_i(\rho u)/(\rho u)_i \quad (4)$$

In Eq. (4) the index i denotes the initial station of the integration on the x-axis. Combining Eqs. (3) and (4) with the axial momentum equation, reduced for $r \rightarrow O$, i.e. $\rho u du = -dp$, there is obtained after integration in the x-direction

$$\Omega^2 = \Omega_i^2 + 2[(p_i - p)\Omega_i^2]/(\rho u_i^2) \quad (5)$$

In order to evaluate Eq. (5), the pressures on the axis of the vortex p_i and p have to be known. For the present study it is assumed that the radial distribution of the azimuthal velocity is given by rigid-body rotation in the core of the vortex, and by a potential vortex outside of it. This distribution, already used by Prandtl in [2], is

$$p_i = p_{i\infty} - \rho\Omega_i^2 R^2 = p_{i\infty}\ \rho w_i^2 max \quad \text{and}$$
$$p = p_\infty - \rho\Omega^2 R^2 = p_{i\infty}\ \rho w_{max}^2 \quad (6)$$

In Eq. (6) $p_{i\infty}$ and p_∞ are the static pressures far away from the axis of the vortex, $p_{i\infty}$ at the initial x-station, and p_∞ at the x-station considered further downstream; the maximum values of the azimuthal velocity are denoted as w_{imax} and w_{max}. The radius R is the radial distance from the axis, where the maximum values of w are attained. It is further assumed that the axial velocity can vary for $r \rightarrow \infty$; then Eq. (5), with the aid of Eq. (6) and the axial velocities $u_{i\infty}$ and u_∞, can be cast into the following form:

$$\Omega/\Omega_i = [1 + (u_\infty^2 - u_{i\infty}^2)/(u_i^2 - 2R^2\Omega_i^2)]^{1/2} \quad (7)$$

If the angular velocity is to vanish, that is if breakdown is to occur, the square-bracketed term on the right-hand side of Eq. (7) has to vanish. This condition is satisfied for

$$R\Omega_i/u_i = \{(1/2)[1 + (u_\infty^2 - u_{i\infty}^2)/u_i^2]\}^{1/2} \quad (8)$$

Eq. (8) without the term $(u_\infty^2 - u_{i\infty}^2)/u_i^2$ was first stated in the form $R\Omega_\iota/u_i = 1/2^{1/2}$ by Billant et al. in [3]. The above considerations show that it can directly be obtained from simplified momentum considerations, extended to variable axial flow far away from the axis of the vortex. It was also stated in [3] that for uniform axial flow for $r \rightarrow \infty$, Eq. (8), in the form $R\Omega_\iota/u_i = 1/2^{1/2}$, agrees well with available experimental data. The critical value of the ratio $R\Omega_i/u_i$ for which breakdown for variable axial flow is to occur is given in Table 1.

Table 1
Critical values of $(R\Omega_i/u_i)$ for breakdown

$(u_\infty{}^2 - (u_{i\infty}{}^2)/u_i{}^2$	−1.0	−0.75	−0.5	−0.25	0.0	0.5	1.0
$(R\Omega_i/u_i)_{crit.}$	0.0	0.354	0.5	0.61	0.71	0.87	1.0

2.3. Solution for compressible inviscid flow

Since the density ρ does not appear in Eq. (2), the azimuthal momentum equation, as given by Eqs. (3) or (4), remains also valid for compressible flow. It is instructive to eliminate the density gradient in Eq. (3) with the aid of the axial momentum equation, the thermal equation of state $\rho = p/RT$, and the energy equation in the form $\rho u c_p dT = u dp$. Introducing the speed of sound $a^2 = p/\rho$, the differential expression $d\Omega/\Omega$ can be written in terms of dp/p in the following form:

$$d\Omega/\Omega = [(Ma^2 - 1)/(\gamma Ma^2)]dp/p \quad (9)$$

The quantity Ma in Eq. (9) is the axial Mach number $Ma = u/a$. It is seen that the above relation is analogous in form to the area-Mach number relation for one-dimensional compressible inviscid flow, given, for example, in [4]: in subsonic flow for $Ma < 1$ the angular velocity increases, $d\Omega > O$, when the pressure decreases, $dp < O$, while in supersonic flow for $Ma > 1$, the angular velocity decreases, $d\Omega < O$, when the pressure decreases, $dp < O$. Note that the behavior of $\Omega_\iota/\Omega_\iota$ is opposite to that of A/A^* in the area-Mach number relation, although both expressions are almost identical in form.

For small entropy changes the isentropic relation for ρ/ρ_0 can be used in Eq. (4), although the flow in the core of the vortex is not isentropic, and Ω/Ω_i can be expressed in terms of the axial Mach number or the pressure ratio p/p_0, similar to the area-Mach number relation. Then the change of Ω/Ω_ι in continuous axial compression, or in discontinuous compression, caused by a normal shock intersecting the vortex can be determined, as was done in [5], in agreement with the experimental data of [6] for the condition that the total temperature in the vortex is constant.

2.4. Viscous effects

If the Stokes stresses are included, the azimuthal momentum equation reads

$$\rho(uw_x + vw_r + vw/r) = r^{-2}\{r^2\mu[r(w/r)_r]\}_r + (\mu w_x)_x \quad (10)$$

For axially symmetric flow Eq. (10) reduces for $r \rightarrow O$ with $w = \Omega r$ to

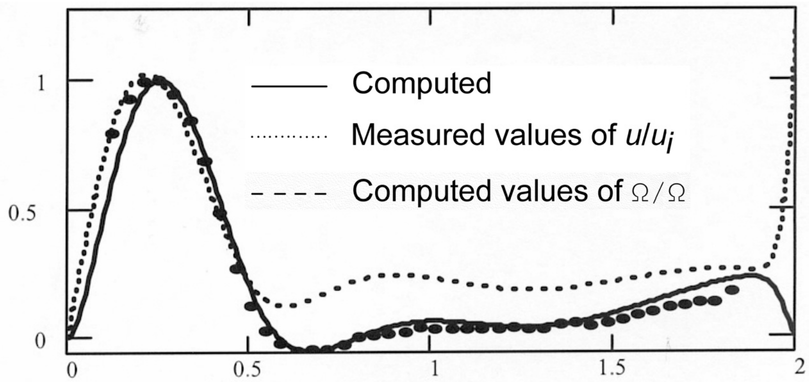

Fig. 1. Comparison of angular velocity and axial velocity component along the axis for $Re = 1560$. Data were provided by Okulov et al. [7].

$$\rho u \Omega_x + 2(\rho v)_r \Omega = \mu_T T_x + \mu \Omega_{xx} \tag{11}$$

If the viscosity μ is assumed to be constant, the second term on the right-hand side of Eq. (11) drops out. With the aid of the continuity equation, Eq. (1), the last equation for $r \to O$ becomes

$$\rho u \Omega_x - (\rho u)_x \Omega = \mu \, \Omega_{xx} \tag{12}$$

When Eq. (12) is multiplied by R and written in the form

$$(R\Omega/\rho u)_x = [\mu R/(\rho u)^2] \, \Omega_{xx} \tag{13}$$

it is seen that the left-hand side of Eq. (13) represents the slope of the ratio ($R\Omega/\rho u$), which was found to be constant for uniform incompressible axial flow u_∞ far away from the axis of the vortex. Eq. (13) shows, however, that the slope of ($R\Omega/\rho u$) cannot vanish, if the viscosity μ is taken into account – in particular when the mass flux ρu tends to zero. Hence the angular velocity may not approach zero when the axial velocity component does.

This behavior of the flow was recently checked by Okulov et al. [7], who investigated the incompressible viscous flow in a cylindrical container with a rotating top in an experiment and with a numerical solution of the Navier–Stokes equations. Figure 1 shows the measured and computed values of the normalized distribution of the axial velocity component u/u_i and the computed distribution of the angular velocity Ω_ι/Ω_r along the axis for a Reynolds number of $Re = 1560$. The results shown confirm the validity of Eq. (4) for the interval $0.0 \leq x \leq 0.5$ of the dimensionless streamwise coordinate x. Further downstream, the measured and computed values of the axial velocity component are close to zero, while the computed angular velocity does not vanish but seems to be shifted by an amount that depends on x, as can be implied from Eqs. (12) or (13). Thus the angular velocity may not always vanish in the immediate vicinity of the stagnation point of the axial flow.

3. Concluding remarks

In the present study, the flow near the axis of slender vortices was investigated with simplified relations. It could be shown that for incompressible inviscid flow the angular velocity is directly proportional to the axial velocity component, and for compressible inviscid flow it is proportional to the axial mass flux. Density variations affect the flow in a manner that depends on whether the axial flow is subsonic or supersonic. The analysis confirmed the breakdown criterion derived by Billant et al. and extended it to variable axial flow far away from the axis of the vortex. The influence of the Stokes stresses was investigated for the immediate vicinity of a stagnation point. The results obtained indicate that the angular velocity does not vanish, when the axial velocity component tends to zero.

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