On higher-order approximation in the MFDM method

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Abstract

Recent developments in a higher-order approximation to both classical finite difference (FD) and meshless finite difference methods (MFDM [1]) are discussed, as well as validation of this approach through the analysis of 1D boundary value problems. The higher-order approximation concept has been introduced in [2] and developed further in [3]. It is based on expansion of the FD operators into Taylor series. The same mesh, as in the case of the lower-order approximation, is used, but selected higher-order terms are included. For boundary value problems of the *m-th* order, the approach provides results that do not depend on the quality of the FD operator used. They are exact within the *2n-th* order Taylor series. In the present study not only smooth solutions, but also jumps of a searched function and its derivatives may be accounted for. Preliminary 1D tests done so far provided very encouraging results.

Keywords: Higher-order approximation; Meshless finite difference method; Jumps

1. Introduction

An approach to apply a higher-order approximation in finite difference analysis of boundary value problems is considered. This approach has been introduced for the first time in [2] and preliminarily developed in [3]. Its further development, with special emphasis laid upon including jumps of the searched function and/or its derivatives, is considered here. Presented also is analysis of a variety of 1D boundary value problems testing the higher-order FD approach proposed. It may be applied in both classical and meshless FD formulations.

The main concept of the higher-order FD formulation is based on the local expansion of FD (MFD) operators into the truncated Taylor series. In the case of an *n-th* order FD operator the $0 \div$ *m-th* order terms of the Taylor series expansion are used to express this FD operator; the terms of *2n-th* order and higher are neglected, while the terms of the $m + 1 \div$ *2m-th* order are used as a higher-order correction term. Such an approach provides a solution exact within the Taylor series of the *2m-th* order, and does not depend on the quality of a FD operator used.

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2. Formulation of the boundary value problem

The approach considered is general and, regardless of the formulation type, may be used to analyse boundary value problems where calculation of derivatives is involved. However, for the sake of simplicity, only the strong (local) formulation is discussed here:

$$\mathsf{L}u = f \qquad \textit{in the domain } \Omega, \textit{ and}$$
$$\mathsf{G}u = g \qquad \textit{on the boundary } \partial\Omega,\ u = u(P) \tag{11}$$

3. Higher-order FD discretization

Having introduced nodes P_i, $i = 1, \ldots, n$ in the domain $\Omega \cup \partial\Omega$, and having selected appropriate FD stars, the given operator L is discretized introducing, at first, a low-order difference operator $L^{(L)}$ at a point P_i:

$$\mathsf{L}u_i \approx L^{(L)}u_i \tag{2}$$

Expanding this operator into the Taylor series at every collocation point P_i, as mentioned above, one gets from Eqs (1) and (2):

$$L^{(L)}u_i = \mathsf{L}u_i + \Delta_i + R_i = f_i + \Delta_i + R_i \approx f_i + \Delta_i \tag{3}$$

after retaining $m + 1 \div 2m$-*th* order correction terms denoted as Δ_i, and neglecting higher-order ones denoted as R_i. The correction term Δ_i includes also jump terms $J^{(k)}$ of the function itself and its derivatives up to *2n-th* order ($k = 0, \dots, 2n$) i.e.

$$\Delta_i = \Delta(u_i^{(n+1)},\ldots,u_i^{(2n)};J^{(0)},\ldots,J_i^{(2n)}) \tag{4}$$

In general, higher-order derivatives $u_i^{(n+1)}, \dots, u_i^{(2n)}$ may be evaluated, e.g., by an appropriate composition of the lower-order ($u_i^{(1)}, \dots, u_i^{(n)}$) derivatives, values of which may be found using the already known lower-order solution $u_i^{(L)}$ of the boundary value problem considered:

$$L^{(L)}u_i^{(L)} = f_i, \quad i = 1,\ldots,n \tag{5}$$

In order to obtain in this way the FD solution of the boundary value problem in question, based on the higher-order approximation, one needs to solve again the same (lowest-order) FD equation (3), though with a modified right-hand side. As mentioned before, such a solution does not depend on the quality of the $L^{(L)}$ FD operator used. Its precision depends only on the truncation error of the Taylor series applied.

It is worth stressing that:

- only the right-hand side of the FD equations changes;
- the final FD solution is obtained in only two steps;
- the final solution will suffer only from the truncation error of the Taylor series and *does not depend on the quality of the FD operator used.*

4. Higher-order discretization of the boundary conditions

Let G be a differential operator given on the boundary. In the general case a separate *2n-th* order polynomial approximation is considered for MFD discretization of the boundary conditions. The approach consists of the following steps:

(i) MFD discretization $Gu_i \approx \mathsf{G}u_i$ of the boundary operator G, followed by the Taylor series expansion

$$Gu_0 = g_0 + \Delta(u_0', u_0'', \ldots, u_0^{(n)}, u_0^{(n+1)}, \ldots u_0^{(2n)}) + R \tag{6}$$

at boundary node P_0.

(ii) Elimination of the lower-order ($1, \dots, n$) derivatives of u_0 using the boundary condition and the equation given in the considered domain but specified on its boundary.

(iii) Replacement of the HO derivatives ($n + 1, \dots, 2n$) of u_0 at boundary nodes by the ones defined at the closest internal nodes in the domain (using the Taylor series expansion).

(iv) Generation of MFD formulas for these HO derivatives, usually by the formula composition approach.

5. Solution of the 1D benchmarks problem: higher-order FDM analysis of beam deflection

Although the benchmark problems presented here are based on the classical FDM, using regular meshes, the whole approach may be applied in the same way to the meshless FDM solution procedure [1] as well. Full automation of this approach is provided in [1]. The approach proposed here has been preliminarily tested on various 1D boundary benchmark problems, including analysis of beam deflection:

$$\mathsf{L}w(x) = \frac{d^2w}{dx^2} \equiv f(x), \quad f(x) = -\frac{M(x)}{EJ} \quad \text{for} \quad x \in [0, L] \tag{7}$$

$$\mathsf{G}w(x) = g(x), \quad \mathsf{G}w = \{w,w'\} \quad \text{for} \quad x \in B \tag{8}$$

The following solution algorithm is proposed:

- Low-order discretization of differential operators described in Eqs (7) and (8):

$$\mathsf{L}(x) \equiv \frac{d^2}{dx^2}, \quad w_i^{II} \approx L^{(L)}w_i = \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2}, \quad i = 1, \ldots, N, \quad h = const$$
$$\mathsf{G}(x) \equiv \{1, \frac{d}{dx}\}, \quad w_i \equiv w_i, \quad w_i^I \approx G^{(L)}w_i = \frac{w_{i+1} - w_{i-1}}{2h}, \quad x_i \in B \tag{9}$$

- Low-order FD solution $u^{(L)}$:

$$\left\{\begin{array}{l} Lu_i = \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} = f_i, \quad i = 1,\ldots,N \\ Gu_i = \frac{w_{i+1} - w_{i-1}}{2h} = g_i, \quad x_i \in B \end{array}\right. \rightarrow u^{(L)} \tag{10}$$

- Correction terms:

$\Delta^{(L)}$ corresponding to the FD operator L
$\Delta^{(G)}$ corresponding to the FD boundary operator G:

$$\Delta_i^{(L)} = \Delta(h, w_i^{IV}, J_i^{(0)}, J_i^{(1)}, J_i^{(2)}, J_i^{(3)}, J_i^{(4)}), \quad i = 1, \ldots, N$$
$$\Delta_i^{(G)} = \Delta(h, w_i^{I}, w_i^{II}, w_i^{III}, w_i^{IV}), \quad x_i \in B$$

where :

$$\begin{array}{ll} \mathrm{J}^{(0)} = \Delta w & \text{(jump in the beam deflection)} \\ \mathrm{J}^{(1)} = \Delta\varphi & \text{(jump in the deflection angle } \varphi(x)) \\ \mathrm{J}^{(2)} = \frac{M}{EJ} & \text{(jump in the bending moment M(x))} \\ \mathrm{J}^{(3)} = \frac{Q}{EJ} & \text{(jump in the shear force Q(x))} \\ \mathrm{J}^{(4)} = \frac{q}{EJ} & \text{(jump in the unform loading q(x))} \end{array} \tag{11}$$

- Evaluation of the higher-order terms may be done in several ways:
  - formula composition: higher-order FD operators are determined using composition of lower-order

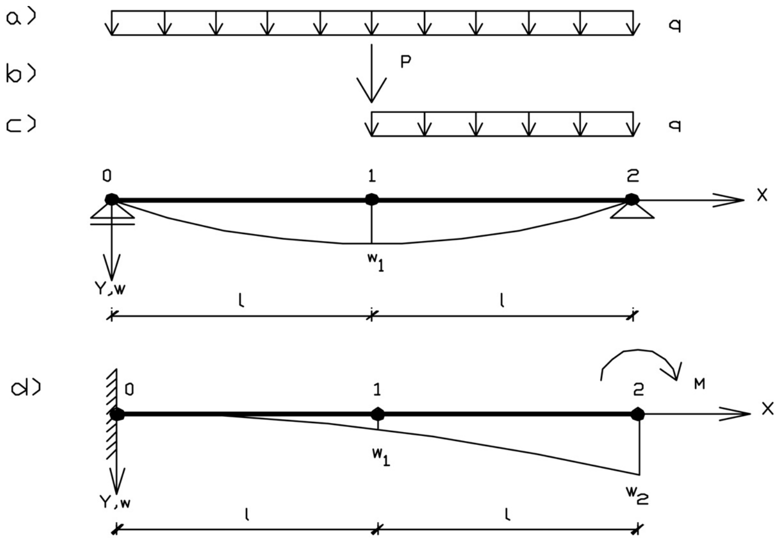

Fig. 1. (a)–(c) Simply supported beam under three types of loading; (d) cantilevered beam.

ones, which are already known from the lower order FD solution;
- *multi-point* approach: using a *multi-point* approach [5], in order to evaluate correction terms Δ_i, seems to be a very promising technique, especially for linear differential operators. Here a combination of the equation's right-hand side values replaces a combination of the lower-order FD formulas in additional nodes. The approach holds in the 2D domain as well, and could be very helpful in the difference analysis of the boundary value problem;
- other techniques (subsequent differentiation of the considered *n-th* order equation).

- Higher-order FD solution $u^{(H)}$ obtained from:

$$\left\{\begin{array}{l} Lu_i = \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} = f_i + \Delta_i^{(L)},\ i = 1,...,N \\ Gu_i = \frac{w_{i+1} - w_{i-1}}{2h} = g_i + \Delta_i^{(G)},\ x_i \in B \end{array}\right. \rightarrow u^{(H)} \tag{12}$$

6. Simple numerical examples

A simply supported beam subjected to three kinds of loads (uniformly distributed and concentrated) as well as the cantilevered beam under concentrated bending moment shown in Fig. 1 have been analysed. The structure was discretized using only three regularly spaced nodes. The low and higher order solutions have been obtained. The correction term Δ_i has been moved

Table 1
Values of the FD operators evaluated at the central node of the beam and relevant correction terms, together with the final FD solution and the exact analytical one

	LO FD solution $w_1^{(L)}$	$(w_1^{II})^{(L)}$	Composite formula w_1^{IV}	Jump $J_1^{(k)}$	Correction term $\Delta_1 = \frac{1}{12}l^2 w_1^{\mathrm{IV}} + \sum_{k=1}^{4} \frac{J_1^{(k)} l^{k-2}}{k!}$	HO FDM $w_1^{(H)}$
(a)	$\frac{1}{4}\frac{ql^4}{EJ}$	$-\frac{1}{2}\frac{ql^2}{EJ}$	$\frac{q}{EJ}$	0	$\frac{1}{12}\frac{ql^2}{EJ}$	$\frac{5}{24}\frac{ql^4}{EJ}$
(b)	$\frac{1}{4}\frac{\mathrm{Pl}^3}{EJ}$	$-\frac{1}{2}\frac{Pl}{EJ}$	0	$J_1^{(3)} = \frac{P}{EJ}$	$\frac{1}{6}\frac{Pl}{EJ}$	$\frac{1}{6}\frac{Pl^3}{EJ}$
(c)	$\frac{1}{8}\frac{ql^4}{EJ}$	$-\frac{1}{4}\frac{ql^2}{EJ}$	0	$J_1^{(4)} = \frac{q}{EJ}$	$\frac{1}{24}\frac{ql^2}{EJ}$	$\frac{5}{48}\frac{ql^4}{EJ}$

Table 2
Values of the FD operators evaluated at two nodes (0), (1) of the cantilever beam and relevant correction terms, together with the final FD solution (which is also the exact analytical one)

Node (i)	FD operator	LO MFD $w_{i+1}^{(L)}$	w_i^{II}	w_i^{III}	w_i^{IV}	$J_i^{(k)}$	*Correction term* Δ_i	HO MFD $w_{i+1}^{(H)}$
$i = 0$	$Gw_0 = \frac{w_1 - w_0}{l}$	0	$\frac{M}{EJ}$	0	0	0	$\Delta_0^{(G)} = \frac{1}{2} l w_0^{II} + \frac{1}{6} l^2 w_0^{III} + \frac{1}{24} l^3 w_0^{IV} = \frac{1}{2}\frac{Ml}{EJ}$	$\frac{1}{2}\frac{Ml^2}{EJ}$
$i = 1$	$Lw_1 = \frac{w_0 - 2w_1 + w_2}{l^2}$	$\frac{Ml^2}{EJ}$	$\frac{M}{EJ}$	0	0	0	$\Delta_1^{(L)} = \frac{1}{12} l^2 w_1^{IV} = 0$	$2\frac{Ml^2}{EJ}$

to the right-hand side of the FD equations. All relevant quantities are shown in Table 1 (simply supported beam) and in Table 2 (cantilevered beam).

Due to the simplicity of the problem, the exact analytical solution has been obtained for each case, solved using the HO FDM.

7. Final remarks

A higher-order approximation approach to FD (or MFD) analysis of boundary value problems involving jumps of the searched function and its derivatives has been discussed. Though the approach is general, only a few linear 1D simple benchmark boundary value problems have been presented here. Application of this approach to 2D and 3D problems will be considered next, especially using the meshless FDM version.

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