Locking-free *hp*-discontinuous Galerkin methods for Timoshenko beams

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Abstract

With the ultimate goal of eliminating a long history of issues that have plagued the structural mechanics community such as the locking phenomenon, we analyze a family of discontinuous Galerkin methods for the Timoshenko beam problem. We prove that the rate of convergence in the energy seminorm is $p + 1/2$ when polynomials of degree p are employed to approximate the unknowns. The estimate is sharp and independent of the thickness-to-length ratio of the beam, which shows that the method is free from shear locking.

Keywords: Shear locking; Timoshenko beams; Discontinuous Galerkin method

1. Introduction

In this paper, we introduce and numerically study discontinuous Galerkin (DG) methods for Timoshenko beams. The Timoshenko beam model, see [1], can be written as: find $w(x)$, $\theta(x)$, $M(x)$ and $T(x)$ satisfying

$$w'(x) = \theta(x) - d^2 T(x), \quad \theta'(x) = M(x),$$
$$M'(x) = T(x), \quad T'(x) = q(x) \tag{1}$$

for all $x \in \Omega = (0, 1)$ and satisfying the boundary conditions

$$w(0) = w_0 \quad w(1) = w_1 \quad \theta(0) = \theta_0 \quad \theta(1) = \theta_1 \tag{2}$$

Here, the unknowns are the transverse displacement w, the rotation of the transverse cross-section of the beam θ, the bending moment M, and the shear force T. The parameter d is proportional to the thickness-to-length ratio of the beam, therefore we assume without loss of generality that $0 < d < 1$. We are particularly interested in the case where $d \ll 1$, because for thin beams the construction of finite element approximation is delicate. Arnold [2] analyzed the continuous version of the finite element method and proved that if no modifications are made then the method exhibits locking. By using the so-called reduced integration technique, he was able to prove locking-free error estimates.

In the present, study we prove and numerically verify that a wide class of DG methods overcomes locking. It is worthwhile to point out that the h version is free from locking even if all the unknowns are approximated by piecewise constant functions. The main reason why these methods are free from locking is the extra flexibility of the approximating functions provided by their discontinuous nature.

2. The DG method and the main result

Let $\mathcal{T} = \{I_j = (x_{j-1}, x_j), j = 1, \ldots, N\}$ be a triangulation of the computational domain $\Omega = (0,1)$. Then, we write

$$(\varphi, \psi)_{\Omega_h} := \sum_{j=1}^{N} (\varphi, \psi)_{I_j} \text{ where } (\varphi, \psi)_{I_j} := \int_{I_j} \varphi(x)\psi(x)\, dx \tag{3}$$

and $\Omega_h = \cup_{j=1,\ldots,N} I_j$. We also write $\langle \mathrm{R}, [\![\varphi n]\!] \rangle_{\mathcal{E}_h} := \sum_{j=0}^{N} R(x_j)\, [\![\varphi n]\!](x_j)$. Here, R is a function defined on the set of nodes $\mathcal{E}_h := \{x_0, x_1, \ldots, x_N\}$. The *jump* of the function φ, $[\![\varphi n]\!]$, is defined as follows. If the node e is in, $\mathcal{E}_h^o := \{x_1, x_2, \ldots, x_{N-1}\} := \{x_1, x_2, \ldots, x_{N-1}\}$, then we take $[\![\varphi n]\!](e) = \varphi(e^+)n_e^+ + \varphi(e^-)n_e^-$,

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where $\varphi(e^{\pm}) := \lim_{\varepsilon \downarrow 0} \varphi(e - \varepsilon n_e^{\pm})$ and $n_e^{\pm} = \mp 1$. For the boundary nodes, we take $[\![\varphi n]\!](0) = -\varphi(0^+)$, $[\![\varphi n]\!](1) = \varphi(1^-)$. We denote by $H^s(\Omega_h)$, the Sobolev spaces of integer orders, and by $\|\cdot\|_{s,\Omega_h}$ the usual *broken* norm in $H^s(\Omega_h)$.

Following [3], the approximate solution $(T_h, M_h, \theta_h, w_h)$ given by the DG method will be sought in the finite dimensional space $[V_h^p]^4$ where $V_h^p := \{v : \Omega_h \to \mathbb{R}: v|I_j \in P^p(I_j),\ j = 1, \ldots, N\}$, and $P^p(K)$ is the set of all polynomials on K of degree not exceeding p. The approximate solution is determined by requiring

$$\begin{aligned}
-(w_h, v_1')_{\Omega_h} &+ \langle \hat{w}_h, [\![v_1 n]\!] \rangle_{\mathcal{E}_h} &&= (\theta_h, v_1)_{\Omega_h} - d^2 (T_h, v_1)_{\Omega_h} \\
-(\theta_h, v_2')_{\Omega_h} &+ \left\langle \hat{\theta}_h, [\![v_2 n]\!] \right\rangle_{\mathcal{E}_h} &&= (M_h, v_2)_{\Omega_h} \\
-(M_h, v_3')_{\Omega_h} &+ \left\langle \hat{M}_h, [\![v_3 n]\!] \right\rangle_{\mathcal{E}_h} &&= (T_h, v_3)_{\Omega_h} \\
-(T_h, v_4')_{\Omega_h} &+ \left\langle \hat{T}_h, [\![v_4 n]\!] \right\rangle_{\mathcal{E}_h} &&= (q, v_4)_{\Omega_h}
\end{aligned} \tag{4}$$

hold for all $v_i \in V_h^p$ for $i = 1, 2, 3, 4$. To complete the definition of the method, we have to define the numerical traces $(\hat{T}_h, \hat{M}_h, \hat{\theta}_h, \hat{w}_h)$ at the nodes. It is through them that the interaction between the degrees of freedom of different intervals is introduced and the boundary conditions are actually imposed. Moreover, their choice is crucial as it affects both the stability and the accuracy of the method. We assume that the form of these traces is as follows. For an interior node $e \in \mathcal{E}_h^o$, we take

$$\begin{aligned}
\hat{w}_h &= \{\!\{w_h\}\!\} + C_{11}[\![w_h n]\!] + C_{12}[\![\theta_h n]\!] + C_{13}[\![M_h n]\!] + \\
&\quad C_{14}[\![T_h n]\!] \\
\hat{\theta}_h &= \{\!\{\theta_h\}\!\} + C_{21}[\![w_h n]\!] + C_{22}[\![\theta_h n]\!] + C_{23}[\![M_h n]\!] + \\
&\quad C_{24}[\![T_h n]\!] \\
\hat{M}_h &= \{\!\{M_h\}\!\} + C_{31}[\![w_h n]\!] + C_{32}[\![\theta_h n]\!] + C_{33}[\![M_h n]\!] + \\
&\quad C_{34}[\![T_h n]\!] \\
\hat{T}_h &= \{\!\{T_h\}\!\} + C_{41}[\![w_h n]\!] + C_{42}[\![\theta_h n]\!] + C_{43}[\![M_h n]\!] + \\
&\quad C_{44}[\![T_h n]\!]
\end{aligned} \tag{5}$$

where $\{\!\{\varphi\}\!\}(e) := \frac{1}{2}(\varphi(e^+) + \varphi(e^-))$. At $x = 0$ and $x = 1$, we take

$$\begin{aligned}
\hat{w}_h(0) &= w_0 \\
\hat{\theta}_h(0) &= \theta_0 \\
\hat{M}_h(0) &= M_h(0^+) + C_{31}(0)\,(w_0 - w_h(0^+)) + \\
&\quad C_{32}(0)\,(\theta_0 - \theta_h(0^+)) \\
\hat{T}_h(0) &= T_h(0^+) + C_{41}(0)\,(w_0 - w_h(0^+)) + \\
&\quad C_{42}(0)\,(\theta_0 - \theta_h(0^+)) \\
\hat{w}_h(1) &= w_1 \\
\hat{\theta}_h(1) &= \theta_1 \\
\hat{M}_h(1) &= M_h(1^-) + C_{31}(1)\,(w_h(1^-) - w_1) + C_{32}(1) \\
&\quad (\theta_h(1^-) - \theta_1) \\
\hat{T}_h(1) &= T_h(1^-) + C_{41}(1)\,(w_h(1^-) - w_1) + \\
&\quad C_{42}(1)\,(\theta_h(1^-) - \theta_1)
\end{aligned} \tag{6}$$

The definition of the DG method is now complete. One can prove that [4] if

$$C_{21} = C_{43},\ -C_{22} = C_{33},\ C_{24} = C_{13},\ C_{31} = C_{42},\ C_{34} = C_{12},\ -C_{11} = C_{44} \tag{7}$$

$$C_{14},\ -C_{23} \geq 0; \quad -C_{32},\ C_{41} > 0 \tag{8}$$

then the method has a unique solution.

We define what we call the *energy seminorm* in which we measure the error of the approximation:

$$|(\varphi_1, \varphi_2, \varphi_3, \varphi_4)|_{\mathcal{A}_h}^2 = \|\varphi_2\|_0 + d^2\|\varphi_1\|_0 + \Theta_{jumps} \tag{9}$$

where

$$\Theta_{jumps} = \sum_{e \in \mathcal{E}_h} \left(C_{14}[\![\varphi_1 n]\!]^2 - C_{23}[\![\varphi_2 n]\!]^2 - C_{32}[\![\varphi_3 n]\!]^2 + C_{41}[\![\varphi_4 n]\!]^2\right)(e) \tag{10}$$

Let us state our error estimate.

Theorem 2.1. *Let $q \in H^s(\Omega_h)$ and let $(T, M, \theta, w) \in H^{s+1}(\Omega_h) \times H^{s+2}(\Omega_h) \times H^{s+3}(\Omega_h) \times H^{s+2}(\Omega_h)$ be the solution of Eqs. (1) and (2). Suppose that the coefficients C_{ij} satisfy Eqs. (7) and (8). Suppose further that C_{14}, C_{23}, C_{32}, C_{41} are $O(1)$.*
Let $(T_h, M_h, \theta_h, w_h) \in [V_h^p]^4$ be the approximate DG solution. Then the following error estimate holds

$$|(e_T, e_M, e_\theta, e_w)|_{\mathcal{A}_h} \leq C \frac{h^{\min(p,s)+1/2}}{\max\{1,p\}^{s+1/2}} \left(\|\theta\|_{s+3,\Omega_h} + \|w\|_{s+2,\Omega_h}\right) \tag{11}$$

for some constant C independent of d, h and p.
Proof. See [3].

Since the constant C is independent of the parameter d, the method is locking-free. There is a positive order of convergence even for piecewise constant approximations. The estimate in the $\mathcal{A}_h$-seminorm is sharp for the h-version of the method because they are actually achieved in our numerical experiments.

3. A numerical experiment

In this section we display some numerical results verifying our theoretical findings. We solve Eq. (1) with $q(x) = \sin \pi x$ and $w_0 = w_1 = \theta_0 = \theta_1 = 0$. Numerical

traces are defined by setting $C_{14}(x) = -C_{23}(x) = -C_{32}(x) = C_{41}(x) = 1$ for all $x \in \mathcal{E}_h$ and the remaining C_{ij}'s are zero at all nodes of the mesh. We display numerical results for $d = 10^{-1}$ through $d = 10^{-8}$. In Table 1 we present the convergence rates of the h version of the the method in the energy seminorm of the error $|\mathbf{e}|_{\mathcal{A}_h}$. The first column shows the polynomial degree p used for approximating all the variables. The second column represents the mesh number, where $mesh = i$ means we used a uniform mesh with 2^i elements in our approximation. The remaining columns indicate the convergence rates for the corresponding error in the energy seminorm. Since the convergence rates are identical for all values of d we deduce that the performance of the method is independent of this parameter, as was proved in Theorem 2.1. From Table 1 we see that the error estimate proved in Theorem 2.1 is sharp in the energy seminorm.

Table 1
Convergence rates for the energy seminorm of the error

p	mesh	$d = 10^{-1}$	$d = 10^{-2}$	$d = 10^{-4}$	$d = 10^{-8}$
	4	0.44	0.44	0.44	0.44
0	5	0.48	0.48	0.48	0.48
	6	0.49	0.49	0.49	0.49
	4	1.57	1.57	1.57	1.57
1	5	1.53	1.53	1.53	1.53
	6	1.52	1.52	1.52	1.52
	4	2.45	2.45	2.45	2.45
2	5	2.48	2.48	2.48	2.48
	6	2.49	2.49	2.49	2.49
	4	3.57	3.57	3.57	3.57
3	5	3.53	3.53	3.53	3.53
	6	3.52	3.52	3.52	3.52
	4	4.45	4.45	4.45	4.45
4	5	4.48	4.48	4.48	4.48
	6	4.49	4.49	4.49	4.49

4. Conclusion

We devised and studied, both theoretically and numerically, a family of DG methods for the Timoshenko beam problem. The method converges to a unique solution even when piecewise constant approximation functions are employed for all the unknowns. Locking-free a priori hp error estimates are proved in the energy seminorm. This estimate is sharp for the family of methods we took into consideration. The error estimate holds independent of the parameter d, which is proportional to the thickness-to-length ratio of the beam. We displayed numerical results for a wide range of this parameter, $10^{-8} \leq d \leq 10^{-1}$. The robustness of these results verify that the method is free from locking effects as predicted by Theorem 2.1.

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